THE OPTIMAL STRATEGY TO RESEARCH PENSION FUNDS IN CHINA BASED ON THE LOSS FUNCTION

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ABSTRACT

Based on the theory of actuarial present value, a pension fund investment goal can be formulated as an objective function. The mean-variance model is extended by defining the objective loss function. Furthermore, using the theory of stochastic optimal control, an optimal investment model is established under the minimum expectation of loss function. In the light of the Hamilton–Jacobi–Bellman (HJB) equation, the analytic solution of the optimal investment strategy problem is derived.

Keywords: Actuarial present value, Pension funds, Fund growth strategy, Investment strategy, Stochastic optimal control, Loss function, Hamilton-Jacobi-Bellman equation

1 INTRODUCTION

Because of the aging of the Chinese population, the gap in China's social pension fund has been increasing rapidly in recent years, and the problems with pension benefits have becoming more and more serious. The investment return of the pension fund is vital to solving the funding gap, so choosing the best investment strategy for managers is becoming more and more important. Effectively investing China's pension fund is a major problem faced by the fund investment institutions.

Researching the pension fund investment primarily involves analyzing the investment strategy. The mean-variance theory proposed by Markowitz in 1952 (Merton, 1969 & 1971) may be regarded as the classic portfolio theory, but the shortcoming of this model is that it only considers a static state and is unable to describe the dynamic character of the question. The time span of pension fund investment is always long, and the investment strategy should be adjusted according to the changes in asset revenue. Finding the optimal portfolio for investment is one of the main problems in investment strategy research.

Recently, many researchers have applied stochastic control theory to the study of the optimal investment strategy for pension funds, which consists of applying dynamic programming theory, setting the objective function, and finding the solution of the HJB equation to get an optimal investment strategy (Cairns, 2000). The main idea of this method is to expand the one-stage model of Markowitz to be multi-stage, transforming the former question to a stochastic linear quadratic control problem by an embedding technique, and to apply the Bellman dynamic programming to find the optimal strategy (Menoncin, 2002). For example, Deelstra et al. (2000) found the closed-form solution of the multi-stage mean-variance model under the Cox-Integral-Ross(CIR, for short) stochastic interest rate, but the objective function of this model only involved the final level of the fund, which did not consider the effect of fund level changes during the control periods of the investment strategy. Haberman and Vigna (2001) considered the expected level of the fund at different stages as a constant and found the optimal investment strategy within the Vasicek framework. However, they ignored the effect of expected level changes on investment strategy. Devolder et al. (2003), Owadally (2003), and Gerrard et al. (2004) considered the optimal investment strategy under the Vasicek and CIR stochastic interest rate methods. Both methods took the expected levels of the fund in different stages as a discrete time series, but they do not mention how to set the expected levels of the fund. Battocchio (2004) and Haberman and Sung (2005) assumed that stock prices follow Geometric Brownian Motion, applied the theory of maximum and backward stochastic differential equations (BSDE), and solved the mean-variance problem of fund investment. They still were not, however, able to set the expected levels of the fund. Setting the expected levels of the fund is the main problem faced by researchers when using the mean-variance method to find the optimal investment strategy.

In this paper we determine the different expected objectives with respect to different relevant investment periods by using the theory of actuarial present value and then provide an objective function by defining a loss function model. We consider issuing treasury bonds to finance the Chinese pension fund. In the light of the loss objective function, we set up an optimal decision model, which minimizes the expected loss during the investing period, and then improved the stochastic control models of Owadally (2003), Gerrard et al. (2004), and Haberman and Sung (2005). Applying the theory of stochastic optimal control, we find the optimal investment strategy by solving the HJB equation.

The paper is organized as follows. In section 2, we determine the different expected investment objectives and provide the loss objective function. In section 3, we set up the optimal decision model, which minimizes the expected loss during the investing period. Section 4 derives an optimal funding policy as a solution of the expected loss minimum problem. Section 5 provides simulation results based on the expected loss minimum model. In the final section, we conclude and comment on future developments.

2 MODEL CONSTRUCTION

When pension managers invest in the securities market, they can choose two kinds of assets (Cf. Haberman & Sung, 2005). One is the riskless asset (bonds) whose rate of return is constant. Its price follows the equation:

$$dx_1(t) = rx_1(t)dt$$

where r is the rate of return. The other is a risky asset (stocks) whose rate of return follows the stochastic differential dynamics:

$$dx_2(t) = \lambda x_2(t)dt + x_2(t)\sigma dw(t)$$

where λ is the mean rate of return of the risky asset, $w(t)$ is Standard Brownian Motion, i.e., $w(t) \sim N(0,t)$, and σ is the volatility of the rate of return. The pension fund gap could be filled by issuing treasury bonds (loans). $\Phi(t)$ is the scale of the issuing treasury bond at time t and the rate of return of the treasury bond is R.

Assume that the proportion of the pension fund $f(t)$ invested in the risky asset is $y(t)$ at time t, and $1 - y(t)$ is the proportion invested in the riskless asset. Then process of accumulating wealth for the pension fund evolves according to the following equation:

$$df(t) = f(t)\left[y(t)(\lambda - r) + r\right]dt - (R - r)\Phi(t)dt + f(t)y(t)\sigma dw(t),\quad f(0) = f$$

Suppose that the pension fund investment horizon is [0, *N*], where *N* is a fixed time, which could be chosen as the peak aging time according to the real situation in China. For instance, if the pension fund investment period chosen is from 2006 to 2025, then *N*=20; the expected target pension fund level F(t) could be determined by the theory of actuarial present value.

Without loss of generality, suppose that there are *n(N, x)* retirees at time *N*, the pension benefit amount of each retiree is b_x, and the average lifetime of the retirees is T-N. Then at time *N*, the actuarial present value of the total amount of pension received by the *n(N, x)* retirees in their future lifetimes is :

$$n(N,x)b_x\int_0^{T-N} e^{-rs}ds = \frac{n(N,x)b_x}{r}\left(1 - e^{-r(T-N)}\right)$$

For the pension payment problem to be settled, the pension fund at time *N* should satisfy the level of $F(N) = \frac{n(N,x)b_x}{r}\left(1 - e^{-r(T-N)}\right)$, and then the expected target of the pension fund level at time *N* could be determined. Applying the theory of actuarial present value, when $t < N$, there should be:

$$F(t) = \frac{n(N,x)b_x}{r}\left(1 - e^{-r(T-N)}\right)e^{-r(N-t)} \tag{1}$$

where $F(t)$ is the expected target value of the pension fund at time *t* ($t < N$).

Definition 1. We define the loss function as $l(t,f) = [F(t) - f(t)]^2 + a[F(t) - f(t)]$, where $F(t) - f(t)$ is the difference between the expected target value and the actual value of the pension fund, which is defined as the loss; *a* ($a \geq 0$) can be viewed as a penalty factor.

Remarks: (1) a can be considered as the risk aversion of the investors, the value of which depends on the risk attitude of the pension investor.

(2) If $F(t) = E(f(t)), a = 1$, then the loss function model is transformed to the mean-variance model.

3 THE LOSS FUNCTION OPTIMIZATION CONTROL PROBLEM

In the investment period [0, *N*], to minimize the expected loss, we could set up the investment decision model:

$$\min_{y(t)} E\left[\int_0^N e^{-js} l(s,f)ds + e^{-jN}K(N,f)\right] \tag{2}$$

s.t $\quad df(t) = f(t)\left[y(t)(\lambda - r) + r\right]dt - [(R - r)\Phi(t)]dt + f(t)y(t)\sigma dw(t)$

$$F(t) = \frac{n(N,x)b_x}{r^2}\left(1 - e^{-r(T-N)}\right)\left(1 - e^{-r(N-t)}\right)$$

$F(0) = f$, where j is the discount factor.

Remarks: $K\left(N,f\right)=\theta\left[\left(F\left(N\right)-f\left(N\right)\right)^2+a\left(F\left(N\right)-f\left(N\right)\right)\right]$, where $\theta\geq 1,\theta$ is specified by the fund investor in light of the degree of importance of the final objective. The larger θ implies that fund investors care more about the loss of the final investment objective.

The purpose of the work reported in this paper is to find the optimal investment strategy y^* (t), which can minimize the present value of the expected loss in the investment horizon [0, N].

Let $H\left(t,f\right)=\min\limits_{y(t)}E\left[\int_t^N e^{-js}l\left(s,f\right)ds+e^{-jN}K\left(N,f\right)\right]$, so

$$H\left(t,f\right)=\min_{y(t)}E\left[\int_t^{t+h}e^{-js}l\left(s,f\right)ds+H\left(t+h,f\right)\right] \tag{3}$$

The boundary condition is

$$H\left(N,f\right)=e^{-jN}K\left(N,f\right)$$

According to the theory of Bellman dynamic programming, we get the HJB equation:

$$\min_{y(t)}\left\{\begin{aligned}&e^{-jt}\left[\left(F\left(t\right)-f\left(t\right)\right)^2+a\left(F\left(t\right)-f\left(t\right)\right)\right]+\frac{\partial H}{\partial t}\\&+\left\{f\left(t\right)\left[y\left(t\right)\left(\lambda-r\right)+r\right]-\left(R-r\right)\Phi(t)\right\}\frac{\partial H}{\partial f}+\frac{1}{2}f^2\left(t\right)y^2\left(t\right)\sigma^2\frac{\partial^2 H}{\partial f^2}\end{aligned}\right\}=0 \tag{4}$$

Let $\varphi\left(y,t,f\right)=e^{-jt}\left[\left(F\left(t\right)-f\left(t\right)\right)^2+a\left(F\left(t\right)-f\left(t\right)\right)\right]+\frac{\partial H}{\partial t}$

$$+\{f\left(t\right)\left[y\left(t\right)\left(\lambda-r\right)+r\right]-\left(R-r\right)\Phi(t)\}\frac{\partial H}{\partial f}+\frac{1}{2}f^2\left(t\right)y^2\left(t\right)\sigma^2\frac{\partial^2 H}{\partial f^2} \tag{5}$$

Suppose that y* (t), is the optimal investment strategy, then:

$$\varphi\left(y^*,t,f\right)=0\,;\qquad \varphi_y'\left(y^*,t,f\right)=0\,;\qquad \varphi_y''\left(y^*,t,f\right)>0 \tag{6}$$

Combining (5) and (6), yields:

$$y^*\left(t\right)=-\frac{\lambda-r}{f\left(t\right)\sigma^2}\times\frac{H_f'}{H_{ff}''} \tag{7}$$

Substituting (7) into (4) gives:

$$e^{-jt}\left[\left(F(t)-f(t)\right)^2+a\left(F(t)-f(t)\right)\right]+\frac{\partial H}{\partial t}+\left[rf(t)-(R-r)\Phi(t)\right]\frac{\partial H}{\partial f}-\frac{1}{2}\left(\frac{\lambda-r}{\sigma}\right)^2\frac{H_f'}{H_{ff}''}=0 \tag{8}$$

If $\partial H/\partial f$ and $\partial^2 H/\partial f^2$ are known, then we can get the closed form of the optimal investment strategy according to (7). The formula (8) is the SDE that the optimal investment strategy of the pension fund satisfies.

4 OPTIMAL INVESTMENT STRATEGY

Let $H(t,f)=e^{-jt}\left[A(t)f^2(t)+B(t)f(t)+C(t)\right]$ (9)

Where $A(t), B(t), C(t)$ are unknown.

From the boundary condition $K(N,f)=\theta\left[\left(F(t)-f(t)\right)^2+a\left(F(N)-f(N)\right)\right]$,

$$\theta e^{-jN}\left[\left(F(N)-f(N)\right)^2+a\left(F(N)-f(N)\right)\right]=e^{-jN}\left[A(N)f^2(N)+B(N)f(N)+C(N)\right]$$

Comparing its coefficients with those in (9), then:

$$A(N)=\theta,\quad B(N)=-\theta\left[2F(N)+a\right],\quad C(N)=\theta\left[F^2(N)+aF(N)\right] \tag{10}$$

Differentiating (9), yields:

$$\frac{\partial H}{\partial t}=-je^{-jt}\left[A(t)f^2(t)+B(t)f(t)+C(t)\right]+e^{-jt}\left[A'(t)f^2(t)+B'(t)f(t)+C'(t)\right] \tag{11}$$

$$\frac{\partial H}{\partial f}=e^{-jt}\left[2A(t)f(t)+B(t)\right] \tag{12}$$

$$\frac{\partial^2 H}{\partial f^2}=2e^{-jt}A(t) \tag{13}$$

From (7), (12), and (13), we can conclude:

$$y^*(t)=-\frac{\lambda-r}{\sigma^2}\left(1+\frac{B(t)}{A(t)f(t)}\right) \tag{14}$$

Substituting (11), (12), and (13) into (8), and letting $\beta=\dfrac{\lambda-r}{\sigma}$, yields:

$$\left[1-jA(t)+2rA(t)+A'(t)-\beta^2A(t)\right]f^2(t)+[B'(t)-2F(t)-a-jB(t)+rB(t)-2(R-r)A(t)\Phi(t)$$

$$-\beta^2B(t)]f(t)+[c'(t)+F^2(t)+aF(t)-jC(t)-(R-r)\Phi(t)B(t)-\beta^2\frac{B^2(t)}{4A(t)}]=0 \tag{15}$$

because (15) is satisfied for every (*t, f*). Then:

$$A'(t)=\left[j+\beta^2-2r\right]A(t)-1 \tag{16}$$

$$B'(t)=\left[j+\beta^2-r\right]B(t)+2F(t)+a+2(R-r)A(t)\Phi(t) \tag{17}$$

$$C'(t)=jC(t)-F^2(t)-aF(t)+(R-r)\Phi(t)B(t)+\beta^2\frac{B^2(t)}{4A(t)} \tag{18}$$

Let $m = j + \beta^2 - 2r$, which together with the boundary condition (10), allows (16) to be rewritten as:

$$A(t) = \left(\theta - \frac{1}{m}\right) e^{-m(N-t)} + \frac{1}{m} \qquad (19)$$

relating to (1), (10), and (19). Solving (17), we get：

$$B(t) = I_1 - I_2 - I_3 \qquad (20)$$

where $I_1 = -\theta[2F(N) + a]e^{-(m+r)(N-t)} + \frac{2F(N)}{m}\left(e^{-(m+r)(N-t)} - e^{-r(N-t)}\right)$；

$$I_2 = 2[\theta - \frac{1}{m}](R-r)e^{-m(N-t)} \int_t^N \Phi(s) e^{r(t-s)} ds + \frac{2(R-r)}{m} \int_t^N \Phi(s) e^{(m+r)(t-s)} ds$$

$$I_3 = \frac{a[e^{-(m+r)(N-t)} - 1]}{m+r}.$$

Then substituting (19) and (20) into (14) yields the explicit solution of the optimal investment strategy:

$$y^*(t) = \frac{-\beta}{\sigma}\left(1 + \frac{m[I_1 - I_2 - I_3]}{\left[(\theta m - 1)e^{-m(N-t)} + 1\right] f(t)}\right) \qquad (21)$$

Remarks: (1) The optimal investment strategy of the pension fund is related to the rate of return of the riskless asset r, the average rate of return of risky asset λ, the amount of the treasury bond $\Phi(t)$, the interest rate of the treasury bond R, the volatility of the risky asset σ, the discount factor j, the objective term N, the expected target level of the fund $F(t)$, and the risk aversion targets of the investors a and θ.

(2) Equation (21) is the closed form solution of the optimal control strategy for pension fund investment.

5 SIMULATION

Assume that T=40, N=20, r= 3%, R = 5%, λ= 10%, σ^2= 12%, j=4%, θ=2, and a=6, then the data of $f(t)$ and $\Phi(t)$ can be referred to the paper by the Labor and Social Security Department (2001)(pages 210-216). According to (21), we can caluclate the optimal investment strategy. The results are shown as follows:

Table 1. Optimal investment strategy for China's pension fund

Years	2000	2001	2002	2003	2004	2005	2006	2007	2008
f(t)(Billion RMB)	94.7	105.4	160.8	220.7	297.5	370.1	488.5	649.7	700.1
$\Phi(t)$ (Billion RMD)	17.0	21.0	40.2	81.6	44.0	55.5	33.6	84.6	102.3
$y^*(t)(\%)$	79.46	68.12	62.32	50.43	48.29	42.51	38.70	34.75	31.29
Years	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2018**	**2020**

F(t) (Billion RMB)	835.6	1042.0	1265.2	1656.9	2081.5	2868.4	3022.1	4781.1	7245.8
$\Phi(t)$ (Billion RMD)	208.9	253.1	274.2	327.0	542.1	684.9	753.9	1283.0	1452.7
$y^*(t)(\%)$	28.64	23.38	20.93	19.02	18.17	15.28	12.83	10.81	7.44

From Table 1, we notice that the optimal investment allocation in the risky asset has a decreasing trend, which means that it seems to decrease as time N approaches. This investment trend satisfies the reality described by Gerrard et al. (2004).

6 CONCLUSIONS

In this paper, we provide the expected objective functions with respect to different relevant investment periods by using the theory of actuarial present value. Then defining a loss function model, we consider issuing treasury bonds to finance the pension fund, which is supported by China's public pension policy. Based on this loss model, we derive an optimal decision model which minimizes the expected loss during the investing period, and then improves stochastic control models. Applying the theory of stochastic optimal control, we conclude the closed form solution of the optimal investment strategy.

There are practical and academic areas in which the dynamic long-term funding approach adapted in this paper could be extended and improved. First, we can introduce time delays in pension valuation (largely due to accounting lags and bias) and extend to include infinite control problems. Second, a stochastically stationary economic and demographic model can be derived as an admissible funding policy.

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