BAYESIAN SHRINKAGE ESTIMATION IN A CLASS OF LIFE TESTING DISTRIBUTION

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ABSTRACT

In the present paper, a class of probability density functions is considered, and the properties of the Bayes estimator and the Bayes Shrinkage estimator of the parameters are studied. The loss functions used are asymmetric loss and squared error loss under different prior distributions. A Bayes estimate of Reliability function is also given.

Keywords: Bayes estimator, Bayes Shrinkage estimator, Squared error loss function (SELF), LINEX loss function (LLF), Reliability function

1 INTRODUCTION

Let us consider that a random variable x follows the distribution presented by a class of probability density functions with the parameter θ and two known positive constants b and c:

$$f(x;\theta,b,c)=\frac{c}{\Gamma b}\left(\frac{x^{bc-1}}{\theta^{b}}\right)\exp\left(-\frac{x^{c}}{\theta}\right);x>0,\theta>0,b,c>0. \quad (1)$$

For different values of constants b and c, the distribution is:

b	c	Distribution
1	1	Negative Exponential Distribution
	1	Two parameter Gamma Distribution
positive integer	1	Erlang Distribution
1		Two parameter Weibull Distribution
1	2	Rayleigh Distribution
3/2	2	Maxwell Distribution

It is now well recognized that the use of the squared error loss function (SELF) in Bayesian estimation may not be appropriate when positive and negative errors have different consequences. To overcome this difficulty, Varian (1975) and Zellner (1986) proposed an asymmetric loss function known as the LINEX loss function (LLF) and its invariant form (Basu & Ebrahimi, 1991) for any parameter θ is given by

$$L(\Delta)=e^{a\Delta}-a\Delta-1\;;\;a\neq 0\;\text{ and }\;\Delta=\left(\frac{\hat{\theta}-\theta}{\theta}\right). \quad (2)$$

The sign and magnitude of 'a' represent the direction and degree of asymmetry respectively. The positive (negative) value of 'a' is used when overestimation is more (less) serious than underestimation. $L(\Delta)$ is the approximate squared error and almost symmetrical if $|a|$ is near to zero.

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A procedure suggested by Thompson (1968) uses the prior information of the parameter in the form of a guess value by shrinking the usual unbiased estimator towards a guess value of the parameter. The values of shrinkage factor k $(0 \le k \le 1)$ are specified by the experimenter according to his belief in a guess value. The shrinkage estimator of a parameter θ, when a prior point guess value θ_0 of θ is available, is given by

$$S = k\hat{\theta} + (1-k)\theta_0, \tag{3}$$

where $\hat{\theta}$ is an estimate of θ.

Bhattacharya (1967) introduced Bayesian ideas into the field of life-testing and reliability analysis. Since then, Soland (1969), Banerjee and Bhattacharya (1979), Zellner (1986), Schabe (1991), Pandey and Rai (1992), Siu and Kelly (1998), Nigm et al. (2003), Stephen (2003), Singh and Saxena (2005), and Son and Oh (2006) have done additional work in this area.

For a given sample $\underline{x} = (x_1, x_2, \ldots, x_n)$ of size n form (1), the likelihood function is given by

$$L(\underline{x};\theta) = g(\theta;T)\,h(\underline{x}), \tag{4}$$

where $g(\theta;T) = \theta^{-nb}e^{-T/\theta}$, $h(\underline{x}) = \left(\frac{c}{\Gamma b}\right)^n \left(\prod_{i=1}^{n} x_i^{bc-1}\right)$ and $T = \sum_{i=1}^{n} x_i^c$.

Here, T is a sufficient statistic for the parameter θ, and the uniformly minimum variance unbiased (UMVU) estimator for θ is $U = \frac{1}{nb}\sum_{i=1}^{n} x_i^c$; whereas $\frac{2nU}{\theta}$ is distributed as a chi-square distribution with $2n$ degrees of freedom. The risk of the UMVU estimator U under the SELF is given as

$$R_{(S)}(U) = E\left(U^2\right) - 2\theta E(U) + \theta^2 = \frac{\theta^2}{n}. \tag{5}$$

Further, the risk under the LLF for estimator U is given as

$$R_{(L)}(U) = E\left(e^{a\Delta'} - a\Delta' - 1\right) = e^{-a}\left(\frac{n}{n-a}\right)^n - 1\ ;\ \Delta' = \left(\frac{U}{\theta} - 1\right). \tag{6}$$

Here the suffixes S and L denote the risk under the SELF and LLF respectively.

There exists a family of conjugate prior distributions that can be obtained by looking at $g(\theta;T)$. The conjugate prior density for the parameter θ can be taken as an inverted Gamma distribution (Raiffa & Schlaifer, 1961) with parameter (α, β) and given by

$$g_1(\theta) = \frac{\beta^\alpha}{\Gamma\alpha}\left(\frac{1}{\theta}\right)^{\alpha+1}\exp\left(-\frac{\beta}{\theta}\right);\theta > 0, \alpha, \beta > 0. \tag{7}$$

The prior mean and prior variance are $\frac{\beta}{\alpha-1};\alpha > 1$ and $\frac{\beta^2}{(\alpha-1)^2(\alpha-2)};\ \alpha > 2$ respectively. For the situation where no prior information about the parameter is available, one may use the uniform, quasi, or improper prior. We consider a class of quasi prior defined as

$$g_2(\theta) = \theta^{-d}\exp\left(-p/\theta\right)\ ;\theta > 0, p, d > 0. \tag{8}$$

The prior mean is $\frac{\Gamma(d-2)}{p^{d-2}}; d \ge 2$, and prior variance is $\frac{\Gamma(d-3)}{p^{2d-4}}\left(p^{d-1} - (d-3)\Gamma(d-2)\right); d > 3$.

This paper suggests some Bayes estimators and Bayes Shrinkage estimators for a family of probability density functions. We study the properties of the suggested estimators in terms of relative efficiencies under two different loss functions. The Bayes estimate of the Reliability function is also obtained.

2 A CLASS FOR THE UMVU ESTIMATOR

The proposed class of estimators for the parameter θ is given as

$$P = l\,U \;;\; l \notin R^{+}. \tag{9}$$

The risks of the estimator P under the SELF and LLF are given as

$$R_{(S)}(P) = E\left(P-\theta\right)^2 = \theta^2\left\{l^2\frac{n+1}{n} + 1 - 2l\right\} \tag{10}$$

and

$$R_{(L)}(P) = E\left\{e^{a\left(\frac{P}{\theta}-1\right)} - a\left(\frac{P}{\theta}-1\right) - 1\right\} = e^{-a}\left(1-\frac{a\,l}{n}\right)^{-n} - a\,l + a - 1. \tag{11}$$

The constant l, which minimizes $R_{(S)}(P)$ and $R_{(L)}(P)$ respectively, is given as

$$l_1 = \frac{n}{n+1} \text{ and } l_2 = \frac{n}{a}\left\{1 - \exp\left(-\frac{a}{n+1}\right)\right\}.$$

The improved estimators in the class (9) are given as

$$P_1 = l_1\,U \qquad \text{and} \qquad P_2 = l_2\,U. \tag{12}$$

In a manner similar to obtaining the risks of P, the risks of these estimators under the SELF and LLF are obtained and are summarized as

$R_{(S)}\left(P_1\right) = \dfrac{\theta^2}{n+1}$	$R_{(L)}\left(P_1\right) = e^{-a}\left(\dfrac{n+1}{n+1-a}\right)^{n} + \dfrac{a}{n+1} - 1$
$R_{(S)}\left(P_2\right) = \theta^2\left\{\dfrac{n+1}{n}l_2^2 + 1 - 2l_2\right\}$	
$R_{(L)}\left(P_2\right) = \left(1+n\right)\left(\exp\left(-\dfrac{a}{n+1}\right) - 1\right) + a$	

3 BAYES SHRINKAGE ESTIMATORS AND THEIR PROPERTIES

The posterior density considering prior density (7) of the parameter θ is given as

$$Z_1(\theta) = \frac{(n\,b\,U+\beta)^{n\,b+\alpha}}{\Gamma(n\,b+\alpha)}\,\frac{e^{-(n\,b\,U+\beta)/\theta}}{\theta^{(n\,b+\alpha+1)}} \;;\; \theta > 0. \tag{13}$$

The Bayes estimator under the SELF for the parameter θ is given as

$$\hat{\theta}_{S1} = E_p(\theta) = \frac{1}{\varphi_1}(n\,b\,U + \beta), \tag{14}$$

where $\varphi_1 = n\,b + \alpha - 1$ and suffix p indicates that the expectation is considered under the posterior density. For utilizing the prior information about the parameter θ in the form of the prior point guess value θ_0, we choose the values of prior parameters α and β such that

$$E(\hat{\theta}_{S1}) = \theta_0 \Rightarrow \beta = \theta_0(\varphi_1 - n\,b). \tag{15}$$

Substituting this value of β in (14), we obtain

$$\overline{\theta}_{S1} = \lambda_1 U + (1-\lambda_1)\,\theta_0 \;;\; \lambda_1 = \frac{n\,b}{\varphi_1}. \tag{16}$$

This is a form of the shrinkage estimator (3), called the Bayes shrinkage estimator. Further, the expectation of $\overline{\theta}_{S1}$ is obtained as

$$E(\overline{\theta}_{S1}) = \theta(\lambda_1(1-\delta)+\delta) \;; \delta = \frac{\theta_0}{\theta}.$$

Similarly,

$$E(\overline{\theta}^2_{S1}) = \theta^2\left\{\lambda_1^2\left(\frac{n+1}{n} + \delta(\delta-2)\right) + \delta^2 + 2\delta\lambda_1(1-\delta)\right\}.$$

Hence, the risk of the Bayes shrinkage estimator $\overline{\theta}_{S1}$ under the SELF is given as

$$R_{(S)}(\overline{\theta}_{S1}) = \theta^2\left\{\lambda_1^2\left(\frac{n+1}{n} + \delta(\delta-2)\right) + (\delta-1)^2(1-2\lambda_1)\right\}. \tag{17}$$

Further, the risk of the Bayes shrinkage estimator $\overline{\theta}_{S1}$ under the LLF is given as

$$R_{(L)}(\overline{\theta}_{S1}) = E\left(e^{a\Delta''} - a\Delta'' - 1\right);\; \Delta'' = \frac{\lambda_1}{\theta}U + \delta(1-\lambda_1) - 1$$

After simplification,

$$R_{(L)}(\overline{\theta}_{S1}) = \left(\frac{n}{n - a\lambda_1}\right)^n e^{\{a(\delta(1-\lambda_1)-1)\}} - 1 + a(1-\delta)(1-\lambda_1). \tag{18}$$

The Bayes estimate of θ under the LLF is obtained by simplifying the equality

$$E_p\left(\frac{1}{\theta}e^{a\hat{\theta}_{L1}/\theta}\right) = e^a E_p\left(\frac{1}{\theta}\right).$$

Therefore, the Bayes estimator for θ under LLF is given as

$$\hat{\theta}_{L1} = \omega_1(n\,b\,U + \beta) \;;\; \omega_1 = \frac{1}{a}\left(1 - \exp\left(\frac{-a}{\varphi_1 + 2}\right)\right). \tag{19}$$

Again,

$$E(\hat{\theta}_{L1}) = \theta_0 \Rightarrow \beta = \left(\frac{1 - n\,b\,\omega_1}{\omega_1}\right).$$

Hence, the Bayes shrinkage estimator under the LLF with this choice of constant is

$$\bar{\theta}_{L1} = \lambda_2 U + (1-\lambda_2)\,\theta_0 \;\; ; \lambda_2 = n\,b\,\omega_1 \;. \qquad (20)$$

The risks of the estimator $\bar{\theta}_{L1}$ under the SELF and LLF are obtained similarly to $\bar{\theta}_{S1}$ and are given as

$$R_{(S)}\left(\bar{\theta}_{L1}\right) = \theta^2 \left\{ \lambda_2^2 \left(\frac{n+1}{n} + \delta(\delta-2) \right) + (\delta-1)^2 \left(1 - 2\lambda_2\right) \right\} \qquad (21)$$

and

$$R_{(L)}\left(\bar{\theta}_{L1}\right) = \left(\frac{n}{n - a\lambda_2} \right)^n e^{\{a(\delta(1-\lambda_2)-1)\}} - 1 + a(1-\delta)(1-\lambda_2). \qquad (22)$$

The expressions of relative efficiency of $\bar{\theta}_{S1}$ with respect to P_1 and $\bar{\theta}_{L1}$ with respect to P_2 under the SELF and LLF are given as

$$RE_{(S)}\left(\bar{\theta}_{S1}, P_1\right) = \frac{R_{(S)}(P_1)}{R_{(S)}\left(\bar{\theta}_{S1}\right)}, \qquad RE_{(L)}\left(\bar{\theta}_{S1}, P_1\right) = \frac{R_{(L)}(P_1)}{R_{(L)}\left(\bar{\theta}_{S1}\right)},$$

$$RE_{(S)}\left(\bar{\theta}_{L1}, P_2\right) = \frac{R_{(S)}(P_2)}{R_{(S)}\left(\bar{\theta}_{L1}\right)} \quad \text{and} \quad RE_{(L)}\left(\bar{\theta}_{L1}, P_2\right) = \frac{R_{(L)}(P_2)}{R_{(L)}\left(\bar{\theta}_{L1}\right)}.$$

The expressions of relative efficiencies are a function of n, a, b, δ, and α. For the selected values of $n =$ $04, 06, 10, 15$; $a = 0.25, 0.50, 1.00$; $b = 1.00, 1.50$; $\delta = 0.50\,(0.25)\,1.50$; $\alpha = 01, 02, 04, 06, 10, 25$, the relative efficiencies have been calculated and presented in Tables 1 – 4. For the risk criterion LLF, the numerical findings are presented only for $b = 1.00$ and $n = 04, 10$.

The Bayes shrinkage estimator $\bar{\theta}_{S1}$ is more efficient than the improved estimator P_1 under the SELF and LLF when $\alpha > 1.00$ for all selected parametric values (Tables 1 and 2). For $\alpha = 1.00$, the relative efficiencies are unaffected by the change in the values of δ, and they are always less than one. The gain in efficiencies decreases as the sample size increases. The maximum value of efficiencies appears at $\delta = 1.00$ for all α in the case of the SELF criterion; whereas in the case of LLF criteria, it occurs only for large $\alpha\,(> 2.00)$. The efficiency decreases with the increases of 'a' when $\delta \geq 1.00$ for all values considered here under the LLF criteria.

The Bayes shrinkage estimator $\bar{\theta}_{L1}$ performs uniformly better than the improved estimator P_2 under both risk criteria for all selected parametric values. The efficiency decreases with increase of the sample size. The maximum efficiency occurs at $\delta = 1.00$. The efficiency tends to decrease with increase in 'a' for $\delta \geq 1.00$ when the risk belongs to the SELF and is in the interval $0.75 \leq \delta \leq 1.25$ when the risk criterion is the LLF.

However, it may be noted that the gain in efficiency around $\delta = 1.00$ tends to increase with an increase in b, but the effective interval becomes smaller for both the Bayes shrinkage estimators.

The posterior density with respect to prior density $g_2(\theta)$ of θ is given as

$$Z_2(\theta) = \frac{(n\,b\,U + p)^{n\,b+d-1}}{\Gamma(n\,b + d - 1)} \, \frac{e^{-(n\,b\,U + p)/\theta}}{\theta^{(n\,b + d)}} \,. \qquad (23)$$

Similarly, the Bayes estimators under the SELF and LLF for the parameter θ are given as

$$\hat{\theta}_{S2} = \frac{1}{\varphi_2}\left(p + n\,b\,U\right) \text{ and } \hat{\theta}_{L2} = \omega_2\left(p + n\,b\,U\right),$$

where $\varphi_2 = n\,b + d - 2$ and $\omega_2 = \frac{1}{a}\left(1 - \exp\left(\frac{-a}{\varphi_2 + 2}\right)\right)$.

Hence, the Bayes shrinkage estimators for the parameter θ are given as

$$\overline{\theta}_{S2} = \lambda_3 U + (1 - \lambda_3)\theta_0 \;;\; \lambda_3 = \frac{n\,b}{\varphi_2} \tag{24}$$

and

$$\overline{\theta}_{L2} = \lambda_4 U + (1 - \lambda_4)\theta_0 \;;\; \lambda_4 = n\,b\,\omega_2 . \tag{25}$$

The risk for these estimators under the SELF and LLF can be easily obtained by making some modifications in expressions (21) and (22). The relative efficiencies for $\overline{\theta}_{S2}$ with respect to P_1 and $\overline{\theta}_{L2}$ with respect to P_2 under the SELF and LLF are given as

$$RE_{(S)}\left(\overline{\theta}_{S2}, P_1\right) = \frac{R_{(S)}(P_1)}{R_{(S)}\left(\overline{\theta}_{S2}\right)}, \qquad RE_{(L)}\left(\overline{\theta}_{S2}, P_1\right) = \frac{R_{(L)}(P_1)}{R_{(L)}\left(\overline{\theta}_{S2}\right)},$$

$$RE_{(S)}\left(\overline{\theta}_{L2}, P_2\right) = \frac{R_{(S)}(P_2)}{R_{(S)}\left(\overline{\theta}_{L2}\right)} \quad \text{and} \quad RE_{(L)}\left(\overline{\theta}_{L2}, P_2\right) = \frac{R_{(L)}(P_2)}{R_{(L)}\left(\overline{\theta}_{L2}\right)} .$$

The expressions of relative efficiencies are a function of n, a, b, δ, and d. For a similar set of selected values as considered above with $d = 2.5, 3.5, 04, 07, 10, 15,$ the relative efficiencies have been calculated and presented in Tables 5 – 8. The numerical findings are presented only for $b = 1.00$ and $n = 04, 10$ when the risk criterion is the LLF.

The estimator $\overline{\theta}_{S2}$ performs better than the improved estimator P_1 in the interval $3.50 \le d \le 10.00$ when the risk criterion is the SELF and for $d \ge 3.50$ when the risk criterion is the LLF (Tables 5 and 6). The efficiency attains maximum at $\delta = 1.00$ for all d when the risk criterion is the SELF; whereas it has maximum only for small $d < 5.00$ when the risk criterion is the LLF. The efficiencies have a decreasing trend with an increase of the sample size for all considered values of parametric space when $d \ge 3.50$. It may be pointed out that the efficiencies decrease with an increase of 'a' for $\delta \ge 1.00$ under the LLF criterion.

The estimator $\overline{\theta}_{L2}$ performs uniformly better than the estimator P_2 under both risk criteria for the parametric space considered here and attains maximum efficiency at $\delta = 1.00$ (Tables 7 and 8). Further, the efficiency decreases as n or 'a' increases for all considered values of δ (except when the risk criterion is the LLF, here efficiency decreases with an increase in 'a' when $\delta \ge 1.00$). In addition, the effect of the increment in b is similar to the Bayes shrinkage estimators $\overline{\theta}_{S1}$ and $\overline{\theta}_{L1}$.

4 BAYES ESTIMATE OF RELIABILITY FUNCTION

The Reliability function $R(t)$ for a specific mission time $t\,(>0)$ is defined as

$$R(t) = P(x > t) = 1 - \frac{1}{\Gamma(b)} \int_0^{t^c/\theta} e^{-z} z^{b-1}\, dz .$$

In particular, for $b = 1, c = 1$, the Reliability function of the Exponential distribution is given as

$$R(t) = \exp(-t/\theta) .$$

In addition, the Hazard rate at the specific mission time $t\,(>0)$ is given as

$$\rho(t) = c\,t^{\,bc-1}\left\{\theta\int_0^{\infty} e^{-S}\,(S\theta + t^{c})^{b-1}\,dS\right\}^{-1}.$$

The Bayes estimate of Reliability function $R(t)$ under the SELF corresponding to the posterior density $Z_1(\theta)$ is obtained as

$$R_{S1}(t) = E_p\big(R(t)\big) = C_1\int_{y=0}^{\infty}\left(1 - \int_0^{y t^c}\frac{e^{-z} z^{b-1}}{\Gamma(b)}dz\right)e^{-(nbU+\beta)y}\,y^{\varphi_1}\,dy$$

$$\Rightarrow R_{S1}(t) = 1 - \frac{C_1}{\Gamma(b)}\int_{y=0}^{\infty}\left(\int_{z=0}^{y t^c} e^{-z} z^{b-1}\,dz\right)e^{-(nbU+\beta)y}\,y^{\varphi_1}\,dy;\quad C_1 = \frac{(nbU+\beta)^{\varphi_1+1}}{\Gamma(\varphi_1+1)}.$$

Similarly, the Bayes estimate of Reliability function $R(t)$ under the SELF corresponding to the posterior density $Z_2(\theta)$ is given as

$$R_{S2}(t) = 1 - \frac{C_2}{\Gamma(b)}\int_{y=0}^{\infty}\left(\int_{z=0}^{y t^c} e^{-z} z^{b-1}\,dz\right)e^{-(nbU+p)y}\,y^{\varphi_2}\,dy;\quad C_2 = \frac{(nbU+p)^{\varphi_2+1}}{\Gamma(\varphi_2+1)}.$$

One may obtain the Bayes estimates of Reliability function $R_{IL1}(t), R_{IL2}(t)$ (say), under the LLF for the given prior densities by solving the given equations

$$\int_{y=0}^{\infty}\frac{1}{J}\left(e^{a} - \exp\left(a\frac{R_{IL1}(t)}{J}\right)\right)e^{-(nbU+\beta)y}\,y^{\varphi_1}\,dy = 0$$

$$\int_{y=0}^{\infty}\frac{1}{J}\left(e^{a} - \exp\left(a\frac{R_{IL2}(t)}{J}\right)\right)e^{-(nbU+p)y}\,y^{\varphi_2}\,dy = 0\ ;\ J = 1 - \int_0^{y t^c}\frac{e^{-z} z^{b-1}}{\Gamma(b)}dz$$

It is not always possible to simplify the above equations to obtain the Bayes estimate of the Reliability function corresponding to the given prior. Therefore, the Bayes estimate of $R(t)$ may be obtained under Varian's (1975) LINEX loss function. The Varian's convex loss function is

$$L\left(\hat{\theta}-\theta\right) = \left\{e^{a\,\hat{\theta}-\theta} - a\left(\hat{\theta}-\theta\right) - 1\right\}\ ;\ a \neq 0,$$

and the Bayes estimates of the Reliability functions $R_{L1}(t), R_{L2}(t)$ (say), are obtained by solving the following equations

$$R_{L1}(t) = -\frac{1}{a}\ln\left\{C_1\int_{y=0}^{\infty}\exp\big(-aJ - (nbU+\beta)y\big)\,y^{\varphi_1}\,dy\right\}$$

$$R_{L2}(t) = -\frac{1}{a}\ln\left\{C_2\int_{y=0}^{\infty}\exp\big(-aJ - (nbU+p)y\big)\,y^{\varphi_2}\,dy\right\}$$

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Table 1. Relative efficiency for the estimator $\overline{\theta}_{S1}$ with respect to P_1 under SELF for different values of n, α and δ

		α					
n	δ	1.00	2.00	4.00	6.00	10.00	25.00
04	0.50	0.8000	1.1765	1.5680	1.5805	1.3938	1.1595
	0.75	0.8000	1.2308	2.1479	2.9124	3.7297	3.9200
	1.00	0.8000	1.2500	2.4500	4.0500	8.4500	39.200
	1.25	0.8000	1.2308	2.1479	2.9124	3.7297	3.9200
	1.50	0.8000	1.1765	1.5680	1.5805	1.3938	1.1595
06	0.50	0.8571	1.1200	1.4026	1.4111	1.2245	1.0857
	0.75	0.8571	1.1546	1.7633	2.2857	2.9056	3.0612
	1.00	0.8571	1.1667	1.9286	2.8810	5.3571	21.429
	1.25	0.8571	1.1546	1.7633	2.2857	2.9056	3.0612
	1.50	0.8571	1.1200	1.4026	1.4111	1.2245	1.0857
10	0.50	0.9091	1.0732	1.2542	1.2587	1.0849	1.0624
	0.75	0.9091	1.0932	1.4545	1.7690	2.1788	2.2846
	1.00	0.9091	1.1000	1.5364	2.0455	3.2818	10.509
	1.25	0.9091	1.0932	1.4545	1.7690	2.1788	2.2846
	1.50	0.9091	1.0732	1.2542	1.2587	1.0849	1.0624
15	0.50	0.9375	1.0492	1.1739	1.1765	1.0213	1.0579
	0.75	0.9375	1.0622	1.3012	1.5094	1.7944	1.8640
	1.00	0.9375	1.0667	1.3500	1.6667	2.4000	6.3375
	1.25	0.9375	1.0622	1.3012	1.5094	1.7944	1.8640
	1.50	0.9375	1.0492	1.1739	1.1765	1.0213	1.0579

Table 2. Relative efficiency for the estimator $\overline{\theta}_{S1}$ with respect to P_1 under LLF for different values of n, 'a', α and δ

			α					
n	a	δ	1.00	2.00	4.00	6.00	10.00	25.00
04	0.25	0.50	0.7527	1.1480	1.5843	1.6115	1.4246	1.0842
		0.75	0.7527	1.1852	2.1228	2.9128	3.7567	3.9464
		1.00	0.7527	1.1883	2.3551	3.9164	8.2224	38.423
		1.25	0.7527	1.1562	2.0208	2.7525	3.5574	3.7807
		1.50	0.7527	1.0941	1.4598	1.4833	1.3191	1.0656
	0.50	0.50	0.7043	1.1191	1.6063	1.6506	1.4633	1.1151
		0.75	0.7043	1.1399	2.1035	2.9255	3.8035	3.9947
		1.00	0.7043	1.1280	2.2690	3.8011	8.0388	37.866
		1.25	0.7043	1.0848	1.9057	2.6113	3.4093	3.6659
		1.50	0.7043	1.0163	1.3625	1.3972	1.2540	1.0591
	1.00	0.50	0.6033	1.0598	1.6719	1.7610	1.5718	1.2010
		0.75	0.6033	1.0497	2.0857	2.9964	3.9716	4.1735
		1.00	0.6033	1.0118	2.1238	3.6286	7.8160	37.504
		1.25	0.6033	1.0056	1.7102	2.3830	3.1854	3.5135
		1.50	0.6033	1.0034	1.1977	1.2563	1.1517	1.0481
10	0.25	0.50	0.8869	1.0611	1.2639	1.2808	1.1105	0.7006
		0.75	0.8869	1.0746	1.4466	1.7733	2.2019	2.3160
		1.00	0.8869	1.0751	1.5058	2.0087	3.2313	10.391
		1.25	0.8869	1.0625	1.4069	1.7073	2.1029	2.2202
		1.50	0.8869	1.0377	1.2014	1.2036	1.0398	0.6556
	0.50	0.50	0.8645	1.0490	1.2750	1.3054	1.1391	0.7209
		0.75	0.8645	1.0562	1.4398	1.7800	2.2298	2.3537
		1.00	0.8645	1.0506	1.4767	1.9748	3.1872	10.299
		1.25	0.8645	1.0325	1.3618	1.6498	2.0334	2.1627
		1.50	0.8645	1.0031	1.1518	1.1525	1.0284	0.6311
	1.00	0.50	0.8189	1.0246	1.3017	1.3636	1.2067	0.7686
		0.75	0.8189	1.0193	1.4295	1.8015	2.3021	2.4507
		1.00	0.8189	1.0022	1.5423	1.9160	3.1185	10.198
		1.25	0.8189	1.0225	1.2786	1.5469	1.9128	2.0681
		1.50	0.8189	1.0011	1.0614	1.0611	1.0255	0.5887

Table 3. Relative efficiency for the estimator $\overline{\theta}_{L1}$ with respect to P_2 under SELF for different values of n, 'a', α and δ

			α					
n	a	δ	1.00	2.00	4.00	6.00	10.00	25.00
		0.50	1.4666	1.5801	1.5798	1.4865	1.3126	1.0419
		0.75	1.7568	2.2079	2.9592	3.4594	3.9082	3.8995
	0.25	1.00	1.8809	2.5450	4.1739	6.2038	11.466	45.486
		1.25	1.7568	2.2079	2.9592	3.4594	3.9082	3.8995
		1.50	1.4666	1.5801	1.5798	1.4865	1.3126	1.0419
		0.50	1.4981	1.5980	1.5859	1.4905	1.3172	1.0480
		0.75	1.8285	2.2782	3.0190	3.5074	3.9419	3.9248
04	0.50	1.00	1.9736	2.6549	4.3203	6.3895	11.739	46.185
		1.25	1.8285	2.2782	3.0190	3.5074	3.9419	3.9248
		1.50	1.4981	1.5980	1.5859	1.4905	1.3172	1.0480
		0.50	1.5742	1.6492	1.6154	1.5153	1.3417	1.0725
		0.75	1.9986	2.4486	3.1744	3.6442	4.0552	4.0212
	1.00	1.00	2.1960	2.9204	4.6799	6.8535	12.449	48.153
		1.25	1.9986	2.4486	3.1744	3.6442	4.0552	4.0212
		1.50	1.5742	1.6492	1.6154	1.5153	1.3417	1.0725
		0.50	1.2025	1.2604	1.2570	1.2764	1.0922	1.0576
		0.75	1.3016	1.4779	1.7887	2.0265	2.2904	2.2559
	0.25	1.00	1.3383	1.5681	2.0824	2.6695	4.0622	11.879
		1.25	1.3016	1.4779	1.7887	2.0265	2.2904	2.2559
		1.50	1.2025	1.2604	1.2570	1.2764	1.0922	1.0576
		0.50	1.2171	1.2690	1.2579	1.1748	1.0907	1.0517
		0.75	1.3293	1.5050	1.8127	2.0465	2.3036	2.2624
10	0.50	1.00	1.3714	1.6044	2.1252	2.7191	4.1261	12.006
		1.25	1.3293	1.5050	1.8127	2.0465	2.3036	2.2624
		1.50	1.2171	1.2690	1.2579	1.1748	1.0907	1.0517
		0.50	1.2529	1.2932	1.2676	1.1793	1.0147	1.0054
		0.75	1.3949	1.5700	1.8733	2.1004	2.3455	2.2911
	1.00	1.00	1.4497	1.6907	2.2282	2.8398	4.2856	12.348
		1.25	1.3949	1.5700	1.8733	2.1004	2.3455	2.2911
		1.50	1.2529	1.2932	1.2676	1.1793	1.0147	1.0054

Table 4. Relative efficiency for the estimator $\overline{\theta}_{L1}$ with respect to P_2 under LLF for different values of n, 'a', α and δ

			α					
n	a	δ	1.00	2.00	4.00	6.00	10.00	25.00
04	0.25	0.50	1.4594	1.5902	1.6031	1.5112	1.3351	1.1611
		0.75	1.7122	2.1735	2.9463	3.4610	3.9192	3.9056
		1.00	1.7920	2.4354	4.0172	5.9925	11.123	44.379
		1.25	1.6425	2.0673	2.7837	3.2703	3.7217	3.7436
		1.50	1.3549	1.4645	2.4760	2.3961	2.2380	1.9838
	0.50	0.50	1.4776	1.6113	1.6250	1.5325	1.3559	1.1815
		0.75	1.7273	2.1965	2.9783	3.4939	3.9454	3.9190
		1.00	1.7781	2.4150	3.9791	5.9302	10.993	43.769
		1.25	1.5851	1.9835	2.6568	3.1189	3.5578	3.6006
		1.50	1.2685	1.3641	2.3766	2.3075	2.1657	1.9297
	1.00	0.50	1.5105	1.6505	1.6661	1.5732	1.3963	1.2217
		0.75	1.7491	2.2352	3.0368	3.5554	3.9948	3.9445
		1.00	1.7543	2.3585	3.8881	5.7923	10.727	42.567
		1.25	1.4570	1.8102	2.4103	2.8305	3.2480	3.3293
		1.50	1.0970	1.1744	2.1930	2.1435	2.0309	1.8284
10	0.25	0.50	1.2003	1.2679	1.2762	1.1991	1.0138	1.0737
		0.75	1.2852	1.4669	1.7892	2.0370	2.3130	2.2807
		1.00	1.3069	1.5332	2.0399	2.6189	3.9933	11.718
		1.25	1.2579	1.4255	1.7217	1.9498	2.2073	2.1878
		1.50	1.1522	1.2039	1.1989	1.1232	1.0949	1.0630
	0.50	0.50	1.2099	1.2812	1.2932	1.2171	1.0314	1.0689
		0.75	1.2919	1.4780	1.8082	2.0617	2.3427	2.3064
		1.00	1.3019	1.5272	2.0317	2.6078	3.9748	11.655
		1.25	1.2350	1.3935	1.6728	1.8880	2.1330	2.1221
		1.50	1.1111	1.1524	1.1401	1.0973	1.0846	1.0627
	1.00	0.50	1.2274	1.3061	1.3259	1.2520	1.0660	1.0272
		0.75	1.3007	1.4960	1.8429	2.1088	2.4004	2.3569
		1.00	1.3839	1.5074	2.0072	2.5777	3.9303	11.518
		1.25	1.1795	1.3215	1.5712	1.7641	1.9882	1.9950
		1.50	1.0217	1.0472	1.0263	1.0807	1.0819	1.0503

Table 5. Relative efficiency for the estimator $\overline{\theta}_{S2}$ with respect to P_1 under SELF for different values of n, d and δ

		d					
n	δ	2.50	3.50	5.00	7.00	10.00	15.00
04	0.50	0.9969	1.3260	1.5680	1.5805	1.4400	1.2497
	0.75	1.0086	1.4611	2.1479	2.9124	3.6000	3.9691
	1.00	1.0125	1.5125	2.4500	4.0500	7.2000	14.450
	1.25	1.0086	1.4611	2.1479	2.9124	3.6000	3.9691
	1.50	0.9969	1.3260	1.5680	1.5805	1.4400	1.2497
06	0.50	0.9956	1.2245	1.4026	1.4111	1.2727	1.0688
	0.75	1.0033	1.3086	1.7633	2.2857	2.8000	3.1137
	1.00	1.0060	1.3393	1.9286	2.8810	4.6667	8.5952
	1.25	1.0033	1.3086	1.7633	2.2857	2.8000	3.1137
	1.50	0.9956	1.2245	1.4026	1.4111	1.2727	1.0688
10	0.50	0.9960	1.1382	1.2542	1.2587	1.1329	0.9204
	0.75	1.0007	1.1856	1.4545	1.7690	2.1039	2.3388
	1.00	1.0023	1.2023	1.5364	2.0455	2.9455	4.8091
	1.25	1.0007	1.1856	1.4545	1.7690	2.1039	2.3388
	1.50	0.9960	1.1382	1.2542	1.2587	1.1329	0.9204
15	0.50	0.9969	1.0934	1.1739	1.1765	1.0665	0.8559
	0.75	1.0000	1.1238	1.3012	1.5094	1.7401	1.9169
	1.00	1.0010	1.1344	1.3500	1.6667	2.2042	3.2667
	1.25	1.0000	1.1238	1.3012	1.5094	1.7401	1.9169
	1.50	0.9969	1.0934	1.1739	1.1765	1.0665	0.8559

Table 6. Relative efficiency for the estimator $\overline{\theta}_{S2}$ with respect to P_1 under LLF for different values of n, 'a', d and δ

			d					
n	a	δ	2.50	3.50	5.00	7.00	10.00	15.00
04	0.25	0.50	0.9571	1.3106	1.5843	1.6115	1.4716	1.2777
		0.75	0.9612	1.4190	2.1228	2.9128	3.6235	4.0002
		1.00	0.9582	1.4430	2.3551	3.9164	6.9980	14.106
		1.25	0.9480	1.3725	2.0208	2.7525	3.4270	3.8071
		1.50	0.9311	1.2309	1.4598	1.4833	1.3611	1.1856
	0.50	0.50	0.9161	1.2962	1.6063	1.6506	1.5113	1.3129
		0.75	0.9133	1.3782	2.1035	2.9255	3.6655	4.0533
		1.00	0.9039	1.3766	2.2690	3.8011	6.8327	13.841
		1.25	0.8883	1.2892	1.9057	2.6113	3.2773	3.6703
		1.50	0.8670	1.1429	1.3625	1.3972	1.2921	1.1300
	1.00	0.50	0.8299	1.2708	1.6719	1.7610	1.6226	1.4113
		0.75	0.8151	1.3010	2.0857	2.9964	3.8197	4.2421
		1.00	0.7953	1.2530	2.1238	3.6286	6.6219	13.577
		1.25	0.7711	1.1381	1.7102	2.3830	3.0477	3.4741
		1.50	0.7433	1.1163	1.1977	1.2563	1.1830	1.0444
10	0.25	0.50	0.9786	1.1320	1.2639	1.2808	1.1588	1.0942
		0.75	0.9802	1.1694	1.4466	1.7733	2.1234	2.3699
		1.00	0.9788	1.1760	1.5958	2.0087	2.8986	4.7431
		1.25	0.9744	1.1506	1.4069	1.7073	2.0299	2.2622
		1.50	0.9671	1.0969	1.2014	1.2036	1.0851	1.0838
	0.50	0.50	0.9609	1.1259	1.2750	1.3054	1.1877	1.1769
		0.75	0.9595	1.1534	1.4398	1.7800	2.1472	2.4071
		1.00	0.9553	1.1503	1.5767	1.9748	2.8571	4.6876
		1.25	0.9482	1.1167	1.3618	1.6498	1.9619	2.1927
		1.50	0.9384	1.0572	1.1518	1.1525	1.0641	1.0502
	1.00	0.50	0.9248	1.1144	1.3017	1.3636	1.2562	1.0296
		0.75	0.9178	1.1223	1.4295	1.8015	2.2095	2.5020
		1.00	0.9083	1.1007	1.5423	1.9160	2.7908	4.6096
		1.25	0.8963	1.0520	1.2786	1.5469	1.8432	2.0744
		1.50	0.8820	1.1023	1.0614	1.0611	1.0634	1.0117

Table 7. Relative efficiency for the estimator $\overline{\theta}_{L2}$ with respect to P_2 under SELF for different values of n, 'a', d and δ

			d					
n	a	δ	2.50	3.50	5.00	7.00	10.00	15.00
04	0.25	0.50	1.5377	1.6000	1.5798	1.4865	1.3500	1.1984
		0.75	1.9864	2.4179	2.9592	3.4594	3.8435	4.0076
		1.00	2.2004	2.9146	4.1739	6.2038	10.004	18.333
		1.25	1.9864	2.4179	2.9592	3.4594	3.8435	4.0076
		1.50	1.5377	1.6000	1.5798	1.4865	1.3500	1.1984
	0.50	0.50	1.5617	1.6133	1.5859	1.4905	1.3544	1.2038
		0.75	2.0579	2.4863	3.0190	3.5074	3.8796	4.0359
		1.00	2.3016	3.0334	4.3203	6.3895	10.253	18.707
		1.25	2.0579	2.4863	3.0190	3.5074	3.8796	4.0359
		1.50	1.5617	1.6133	1.5859	1.4905	1.3544	1.2038
	1.50	0.50	1.6240	1.6561	1.6154	1.5153	1.3788	1.2287
		0.75	2.2291	2.6542	3.1744	3.6442	3.9970	4.1397
		1.00	2.5453	3.3215	4.6799	6.8535	10.893	19.688
		1.25	2.2291	2.6542	3.1744	3.6442	3.9970	4.1397
		1.50	1.6240	1.6561	1.6154	1.5153	1.3788	1.2287
10	0.25	0.50	1.2377	1.2719	1.2570	1.1764	1.1347	1.0952
		0.75	1.3910	1.5616	1.7887	2.0265	2.2470	2.3649
		1.00	1.4509	1.6899	2.0824	2.6695	3.6867	5.7462
		1.25	1.3910	1.5616	1.7887	2.0265	2.2470	2.3649
		1.50	1.2377	1.2719	1.2570	1.1764	1.1347	1.0952
	0.50	0.50	1.2492	1.2781	1.2579	1.1848	1.1733	1.1048
		0.75	1.4185	1.5881	1.8127	2.0465	2.2616	2.3742
		1.00	1.4856	1.7278	2.1252	2.7191	3.7469	5.8255
		1.25	1.4185	1.5881	1.8127	2.0465	2.2616	2.3742
		1.50	1.2492	1.2781	1.2579	1.1848	1.1733	1.1048
	1.50	0.50	1.2788	1.2975	1.2676	1.1993	1.1867	1.1186
		0.75	1.4840	1.6523	1.8733	2.1004	2.3061	2.4090
		1.00	1.5679	1.8181	2.2282	2.8398	3.8964	6.0279
		1.25	1.4840	1.6523	1.8733	2.1004	2.3061	2.4090
		1.50	1.2788	1.2975	1.2676	1.1993	1.1867	1.1186

Table 8. Relative efficiency for the estimator $\overline{\theta}_{L2}$ with respect to P_2 under LLF for different values of n, 'a', d and δ

			d					
n	a	δ	2.50	3.50	5.00	7.00	10.00	15.00
04	0.25	0.50	1.5403	1.6156	1.6031	1.5112	1.3731	1.2193
		0.75	1.9466	2.3892	2.9463	3.4610	3.8536	4.0181
		1.00	2.1014	2.7940	4.0172	5.9925	9.6931	17.828
		1.25	1.8583	2.2663	2.7837	3.2703	3.6547	3.8318
		1.50	1.4225	1.4861	1.4760	1.3961	1.2725	1.1316
	0.50	0.50	1.5604	1.6373	1.6250	1.5325	1.3940	1.2399
		0.75	1.9658	2.4153	2.9783	3.4939	3.8817	4.0381
		1.00	2.0845	2.7698	3.9791	5.9302	9.5825	17.645
		1.25	1.7875	2.1702	2.6568	3.1189	3.4912	3.6724
		1.50	1.3272	1.3835	1.3766	1.3075	1.1970	1.0679
	1.50	0.50	1.5972	1.6777	1.6661	1.5732	1.4344	1.2802
		0.75	1.9966	2.4609	3.0368	3.5554	3.9347	4.0761
		1.00	2.0347	2.7058	3.8881	5.7923	9.3511	17.156
		1.25	1.6365	1.9759	2.4103	2.8305	3.1822	3.3711
		1.50	1.1439	1.1918	1.1930	1.1435	1.0566	1.0488
10	0.25	0.50	1.2407	1.2835	1.2762	1.1991	1.1570	1.0746
		0.75	1.3773	1.5535	1.7892	2.0370	2.2675	2.3913
		1.00	1.6418	1.6531	2.0399	2.6189	3.6226	5.6564
		1.25	1.3429	1.5052	1.7217	1.9498	2.1642	2.2842
		1.50	1.1837	1.2140	1.2989	1.2232	1.1898	1.1194
	0.50	0.50	1.2522	1.2980	1.2932	1.2171	1.0747	1.0614
		0.75	1.3861	1.5666	1.8082	2.0617	2.2966	2.4209
		1.00	1.6323	1.6466	2.0317	2.6078	3.6062	5.6285
		1.25	1.3154	1.4686	1.6728	1.8880	2.0917	2.2088
		1.50	1.1368	1.1591	1.2401	1.1673	1.0623	1.0323
	1.50	0.50	1.2737	1.3257	1.3259	1.2520	1.1096	1.0248
		0.75	1.3996	1.5891	1.8429	2.1088	2.3530	2.4790
		1.00	1.5934	1.6257	2.0072	2.5777	3.5657	5.5652
		1.25	1.2516	1.3887	1.5712	1.7641	1.9497	2.0628
		1.50	1.0384	1.0493	1.1263	1.0607	1.0519	1.0015

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