

A NOTE ON BAYESIAN ESTIMATION OF THE TRAFFIC INTENSITY IN M/M/1 QUEUE AND QUEUE CHARACTERISTICS UNDER QUADRATIC LOSS FUNCTION

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ABSTRACT

Bayes' estimators of the traffic intensity ρ and various queue characteristics in an M/M/1 queue have been derived under the assumptions of different priors for ρ and the quadratic error loss function (QELF). Finally, a numerical example is given to illustrate the results

Key Words: Bayes' estimator, Hyper parameter, M/M/1 queue, Quadratic error loss function, Traffic intensity

1 INTRODUCTION

Statistical analysis is an integral part of formulating a mathematical model for a real system. A model is not of much use unless it is related to the system through empirical data analyses, parameter estimation, and tests of relevant hypotheses. In queuing theory, statistical analyses and Bayesian analyses have taken a back seat and received comparatively less attention. Harischandra and Subba Rao (1988) discussed some problems of classical inference for the traffic intensity ρ involved in an M/E_k/1 queue. Bhattacharyya and Singh (1994) obtained the Bayes' estimator of the traffic intensity ρ for an M/E_k/1 queue under two prior densities. Sharma and Kumar (1999) discussed classical and Bayesian estimators of various characteristics of a M/M/1 queue under squared error loss function. Mukherjee and Chowdhury (2005) obtained the Bayes' estimator of the traffic intensity ρ in an M/M/1 queue under squared error loss function and LINEX loss function. Dey (2006) obtained Bayes estimators of the traffic intensity and various queue characteristics under the assumptions of different priors and the usual squared error loss function.

The objective of this paper is to present the Bayesian estimation for the M/M/1 queue. To be specific, we have obtained the Bayesian estimates of the traffic intensity ρ and various queues' characteristics in an M/M/1 queue such as expected queue length, expected length of waiting line, and the probability of minimum queue size, under the different priors for the queuing parameter ρ under the quadratic error loss function (QELF).

2 PRELIMINARIES

Let us consider an M/M/1 queuing system with the mean arrival rate λ and mean service time $1/\mu$. The analysis for such a queue is now folk lore in the queuing literature, and we know that the random variable X representing the number of customers in the system under steady state has the distribution specified by the probability mass function (pmf) (cf. Kleinrock, 1975, page 96).

$$P(x|\rho) = (1 - \rho) \rho^x, \quad x = 0, 1, 2, 3, \dots \quad (2.1)$$

$$\text{where } \rho = \frac{\lambda}{\mu}, \quad 0 < \rho < 1 \quad (2.2)$$

represents the traffic intensity for the given queuing system.

Based on the pmf in (2.1), various important characteristics of the system are:

- (i) $L_s = \frac{\rho}{1-\rho}$: Expected queue length.
- (ii) $L_q = \frac{\rho^2}{1-\rho}$: Expected length of waiting line.
- (iii) $Q_m = P(X \geq n_0) = \rho^{n_0}$: The probability of minimum queue size n_0 .

The likelihood function (LF) corresponding to the pmf (2.1) is given by:

$$L(\rho) = (1 - \rho)^n \rho^T \tag{2.3}$$

where $T = \sum_{i=0}^{\alpha} x_i$ (2.4)

The expression (2.3) plays a key role in obtaining the Bayes' estimator of ρ under the assumption of the quadratic loss function. In the Bayesian approach, we assume further that some prior knowledge about the queuing parameter ρ is available to the investigator from past experiences with the underlying queuing system. This prior knowledge can often be summarized in terms of the so-called prior densities on the parameter space of ρ . When nothing is known regarding the parameter (ρ) from the past experience, however, this prior of ignorance may be represented by diffuse prior(s) or vague prior(s). For the problem at hand, we assume the following priors:

(1) Diffuse Prior: $g_1(\rho) \propto \frac{1}{\rho^c}, c > 0$ (2.5)

(2) Beta Prior: $g_2(\rho) = \frac{1}{B(a,b)} \rho^{a-1} (1-\rho)^{b-1}, 0 < \rho < 1, a, b > 0$ (2.6)

3 BAYES' ESTIMATOR OF ρ

In this section, the Bayes' estimator of ρ , its posterior variance, and various queue characteristics of an M/M/1 queue are found out using the above prior distribution.

3.1 Bayes' Estimator Under Prior $g_1(\rho)$

We consider the case when the prior density of ρ is $g_1(\rho)$. The LF (2.3) is combined with the prior density (2.5) by using the Bayes' theorem to obtain the so-called posterior density:

$$H_1(\rho | x) = \frac{(1 - \rho)^n \rho^{T-c}}{\int_0^1 (1 - \rho)^n \rho^{T-c} d\rho} \tag{3.1.1}$$

With a quadratic loss function, the Bayes' estimator for ρ with posterior density (3.1) comes out as:

$$\begin{aligned} \rho_1^* &= \frac{\int_0^1 \frac{1}{\rho} H_1(\rho | x) d\rho}{\int_0^1 \frac{1}{\rho^2} H_1(\rho | x) d\rho} \\ &= \frac{T - c - 1}{T + n - c} \end{aligned} \quad (3.1.2)$$

The corresponding posterior variance of ρ is given by:

$$\begin{aligned} V(\rho_1) &= \int_0^1 \rho^2 H_1(\rho | x) d\rho - (\rho_1^*)^2 \\ &= \frac{(T - c + 2)(T - c + 1)}{(T + n - c + 3)(T + n - c + 2)} - \left(\frac{T - c + 1}{T + n - c + 2} \right)^2 \end{aligned} \quad (3.1.3)$$

Also, the Bayes' estimator of L_S , say $L_{S_1}^*$, is

$$\begin{aligned} L_{S_1}^* &= \frac{\int_0^1 \frac{(1 - \rho)}{\rho} H_1(\rho | x) d\rho}{\int_0^1 \frac{(1 - \rho)^2}{\rho^2} H_1(\rho | x) d\rho} \\ &= \frac{(T - c - 1)}{(n + 2)} \end{aligned} \quad (3.1.4)$$

Similarly, the Bayes' estimator of L_q , say $L_{q_1}^*$, is

$$\begin{aligned} L_{q_1}^* &= \frac{\int_0^1 \frac{(1 - \rho)}{\rho^2} H_1(\rho | x) d\rho}{\int_0^1 \frac{(1 - \rho)^2}{\rho^4} H_1(\rho | x) d\rho} \\ &= \frac{(T - c - 2)(T - c - 3)}{(T + n - c)(n + 2)} \end{aligned} \quad (3.1.5)$$

Finally, the Bayes' estimator of Q_m , say $Q_{m_1}^*$, is

$$Q_{m_1}^* = \frac{\int_0^1 \frac{1}{\rho^{n_0}} H_1(\rho | x) d\rho}{\int_0^1 \left(\frac{1}{\rho^{n_0}} \right)^2 H_1(\rho | x) d\rho}$$

$$= \frac{(T - c - n_0)! (T + n - c - 2n_0 + 1)!}{(T + n - c - n_0 + 1)! (T - c - 2n_0)!} \quad (3.1.6)$$

We note that for n sufficiently large as compared to 'c', $\rho_{c_0}^*$, $\rho_{c_1}^*$, $\rho_{c_2}^*$ are all numerically very close to each other. Moreover, we can show that $E[(\rho_{c_i}^* - \rho_{c_j}^*)^2] \rightarrow 0$ as $n \rightarrow \infty$ for any fixed value of c_i and c_j ,

$i \neq j = 0, 1, 2$. By Tchebyshev's inequality, it follows that for any $\varepsilon > 0$, $\lim_{n \rightarrow \infty} \Pr\{(\rho_{c_i}^* - \rho_{c_j}^*)^2 < \varepsilon\} \rightarrow 1$,

showing that in large samples the choice of the constant 'c' (i.e., the choice of the prior) is not very crucial.

3.2 Bayes' Estimator Under Prior $g_2(\rho)$

In this subsection, we consider the case when the prior density of ρ is $g_2(\rho)$. The posterior distribution of ρ can be obtained by using Bayes' theorem as:

$$H_2(\rho | x) = \frac{(1 - \rho)^{n+b-1} \rho^{T+a-1}}{\int_0^1 (1 - \rho)^{n+b-1} \rho^{T+a-1} d\rho} \quad (3.2.1)$$

With a quadratic error loss function, the Bayes' estimator for ρ with posterior density (3.2.1) comes out as:

$$\begin{aligned} \rho_2^* &= \frac{\int_0^1 \frac{1}{\rho} H_2(\rho | x) d\rho}{\int_0^1 \frac{1}{\rho^2} H_2(\rho | x) d\rho} \\ &= \frac{T + a - 2}{T + n + a + b - 2} \end{aligned} \quad (3.2.2)$$

The corresponding posterior variance of ρ is given by:

$$\begin{aligned} V(\rho_2) &= \int_0^1 \rho^2 H_2(\rho | x) d\rho - (\rho_2^*)^2 \\ &= \frac{(T + a + 2)(T + a + 1)}{(T + n + a + b + 1)(T + n + a + b)} - \left(\frac{T + a}{T + n + a + b} \right)^2 \end{aligned} \quad (3.2.3)$$

Also, the Bayes' estimator of L_S , say, $L_{S_2}^*$, is

$$\begin{aligned}
L_{S_2}^* &= \frac{\int_0^1 \frac{(1-\rho)}{\rho} H_2(\rho | x) d\rho}{\int_0^1 \frac{(1-\rho)^2}{\rho^2} H_2(\rho | x) d\rho} \\
&= \frac{(T+a+2)}{(n+b+1)}
\end{aligned} \tag{3.2.4}$$

Similarly, the Bayes' estimator of L_q , say $L_{q_2}^*$, is

$$\begin{aligned}
L_{q_2}^* &= \frac{\int_0^1 \frac{(1-\rho)}{\rho^2} H_2(\rho | x) d\rho}{\int_0^1 \frac{(1-\rho)^2}{\rho^4} H_2(\rho | x) d\rho} \\
&= \frac{(T+a-3)(T+a-4)}{(T+n+a+b-2)(n+b+1)}
\end{aligned} \tag{3.2.5}$$

Finally, the Bayes' estimator of Q_m , say $Q_{m_2}^*$, is

$$\begin{aligned}
Q_{m_2}^* &= \frac{\int_0^1 \frac{1}{\rho^{n_0}} H_2(\rho | x) d\rho}{\int_0^1 \left(\frac{1}{\rho^{n_0}}\right)^2 H_2(\rho | x) d\rho} \\
&= \frac{(T+a-n_0-1)!(T+n+a+b-2n_0-1)!}{(T+n+a+b-n_0-1)!(T+a-2n_0-1)!}
\end{aligned} \tag{3.2.6}$$

4 NUMERICAL RESULTS

We generated 5000 samples of size $n = 50$ from the geometric distribution as mentioned in section (2). Table 1 shows Bayesian estimates of the parameter and posterior variance of ρ for different values of the parameter ρ under the prior distributions (i) diffuse prior density (ii) beta prior density and. Table 2 and Table3 show Bayesian estimates of the expected queue length (L_s), expected length of waiting line (L_q), and the probability of minimum queue size (Q_m) for different values of the traffic intensity (ρ).

Table 1. The Bayesian estimates of the traffic intensity parameter and posterior variance of ρ with $\rho = 0.5$ and $\rho = 0.8$

n	ρ	a	b	ρ_1^* c=0	ρ_1^* c=1	ρ_1^* c=2	$v(\rho_1^*)$ c=0	$v(\rho_1^*)$ c=1	$v(\rho_1^*)$ c=2	ρ_2^*	$v(\rho_2^*)$
50	0.5	1	1	.5778	.5703	.5695	.0019	.0020	.0020	.5612	.0115
		1	2	.5778	.5703	.5695	.0019	.0020	.0020	.5603	.0113
		1	3	.5778	.5703	.5695	.0019	.0020	.0020	.5572	.0111
		2	1	.5778	.5703	.5695	.0019	.0020	.0020	.5501	.0114
		3	1	.5778	.5703	.5695	.0019	.0020	.0020	.5370	.0113
		1	10	.5778	.5703	.5695	.0019	.0020	.0020	.5190	.0102
		10	1	.5778	.5703	.5695	.0019	.0020	.0020	.4650	.0109
50	0.8	1	1	.8112	.8078	.8072	.00056	.00058	.00058	.8038	.0065
		1	2	.8112	.8078	.8072	.00056	.00058	.00058	.8022	.0064
		1	3	.8112	.8078	.8072	.00056	.00058	.00058	.7988	.0064
		2	1	.8112	.8078	.8072	.00056	.00058	.00058	.7962	.0065
		3	1	.8112	.8078	.8072	.00056	.00058	.00058	.7947	.0064
		1	10	.8112	.8078	.8072	.00056	.00058	.00058	.7805	.0061
		10	1	.8112	.8078	.8072	.00056	.00058	.00058	.7491	.0063

Table 2. The Bayesian estimates of L_S , L_q , and Q_m with $\rho = 0.5$ for different prior distributions

n	a	b	T	n_0	L_{S1}^* c=0	L_{S1}^* c=1	L_{S1}^* c=2	L_{S2}^*	L_{q1}^* c=0	L_{q1}^* c=1	L_{q1}^* c=2	L_{q2}^*	Q_{m1}^* c=0	Q_{m1}^* c=1	Q_{m1}^* c=2	Q_{m2}^*
50	1	1	62	3	1.17	1.15	1.13	1.17	0.61	0.59	0.58	.61	.151	.147	.143	.151
50	1	2	62	3	1.17	1.15	1.13	1.15	0.61	0.59	0.58	.59	.151	.147	.143	.146
50	1	3	62	3	1.17	1.15	1.13	1.13	0.61	0.59	0.58	.57	.151	.147	.143	.143
50	2	1	62	3	1.17	1.15	1.13	1.19	0.61	0.59	0.58	.62	.151	.147	.143	.154
50	3	1	62	3	1.17	1.15	1.13	1.21	0.61	0.59	0.58	.64	.151	.147	.143	.158
50	1	10	62	3	1.17	1.15	1.13	1.00	0.61	0.59	0.58	.48	.151	.147	.143	.119
50	10	1	62	3	1.17	1.15	1.13	1.35	0.61	0.59	0.58	.75	.151	.147	.143	.183

Table 3. The Bayesian estimates of L_S , L_q , and Q_m with $\rho = 0.8$ for different prior distributions

n	a	b	T	n_0	L_{S1}^* c=0	L_{S1}^* c=1	L_{S1}^* c=2	L_{S2}^*	L_{q1}^* c=0	L_{q1}^* c=1	L_{q1}^* c=2	L_{q2}^*	Q_{m1}^* c=0	Q_{m1}^* c=1	Q_{m1}^* c=2	Q_{m2}^*
50	1	1	246	3	4.71	4.69	4.67	4.71	3.85	3.83	3.81	3.85	0.563	0.562	0.561	0.563
50	1	2	246	3	4.71	4.69	4.67	4.62	3.85	3.83	3.81	3.77	0.563	0.562	0.561	0.558
50	1	3	246	3	4.71	4.69	4.67	4.54	3.85	3.83	3.81	3.68	0.563	0.562	0.561	0.552
50	2	1	246	3	4.71	4.69	4.67	4.73	3.85	3.83	3.81	3.87	0.563	0.562	0.561	0.565
50	3	1	246	3	4.71	4.69	4.67	4.75	3.85	3.83	3.81	3.89	0.563	0.562	0.561	0.567
50	1	10	246	3	4.71	4.69	4.67	4.02	3.85	3.83	3.81	3.19	0.563	0.562	0.561	0.575
50	10	1	246	3	4.71	4.69	4.67	4.88	3.85	3.83	3.81	4.02	0.563	0.562	0.561	0.574

5 CONCLUSION

It has been observed from Table 1 that there is very little change in values of estimators of ρ and posterior variance (or Bayes' risk) due to change in values of 'c.' When the value of ρ tends to 1, however, the proposed Bayes' estimators approach the true value of the parameter, and in that case, the Bayes' risk is also minimum in comparison to other values of ρ .

In case of beta prior, we have considered seven sets of values of 'a' and 'b' for the analysis:

- 1) When the difference between 'a' and 'b' is large, e.g., we have taken $a=1$, $b=10$.
- 2) When 'a' and 'b' are equal, e.g., $a=b=1$, even if we take $a=b$ =other than 1, we find almost the same results.
- 3) When 'a' is greater than 'b', e.g., $a = 10$, $b = 1$.

From Table 1, it is clear that the difference between estimator and the true value is least in case of (1).

It is interesting to note here that the Bayes' estimator under quadratic error loss function (BQELF) with $c = 0$ has the smallest posterior variance among all, though posterior variances for $c = 1, 2$ are close to it. It also has to be noted that the BQELF performs better than the Bayes' estimator under squared error loss function (SQELF).

The above remarks are just observations based on the data referred to and should be viewed as such. Extensive simulation studies with different sets of sample size and parameter need to be worked out to examine more closely the behavior of the Bayes' estimators ρ and different characteristics of the system under different priors and loss functions.

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