MINIMAX ESTIMATION OF THE PARAMETER OF THE RAYLEIGH DISTRIBUTION UNDER QUADRATIC LOSS FUNCTION

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ABSTRACT

This paper is concerned with the problem of finding the minimax estimator of the parameter θ of the Rayleigh distribution for quadratic loss function by applying the theorem of Lehmann (1950).

Keywords: Bayes estimator; Maximum likelihood estimator; Mean squared error; Minimaxity; Quadratic loss function, Modified Linear-Exponential (MLINEX) Loss function

1 INTRODUCTION

In an expository paper, Siddique (1962) discussed the origin and properties of the Rayleigh distribution. Polovko (1968) and Dyer and Whisenand (1973) noted the importance of this distribution in electro vacuum devices and communication engineering. Dey and Das (2007) obtained Bayesian predictive intervals of the parameter of Rayleigh distribution. The probability density function of the Rayleigh distribution is given by:

$$f(x|\,\theta) = \frac{x}{\theta^2}\, e^{-\frac{x^2}{2\theta^2}} \tag{1.1}$$

where θ is the parameter of the distribution.

Podder et al. (2004) studied the minimax estimator of the parameter of the Pareto distribution under Quadratic and MLINEX loss functions. In this paper, we shall estimate the parameter of Rayleigh distribution by using the technique of minimax approach under quadratic loss function, which is essentially a Bayesian approach. The most important element in the minimax approach is the specification of a distribution function on the parameter space, which is called *prior distribution*. In addition to the prior distribution, the minimax estimator for a particular model depends strongly on the loss function assumed. The basic difference between the philosophy of the minimax and classical estimation is that in minimax estimation the parameter of the distribution is assumed to be a random variable, where as classical estimation regards it as a fixed point.

In this paper, we derive the minimax estimator of the parameter θ of the Rayleigh distribution under quadratic loss function. The derivation depends primarily on Lehmann's Theorem (Lehmann, 1950) and can be stated as follows.

Theorem: Let τ = {Fθ; θεΘ} be a family of distribution functions and D a class of estimators of θ. Suppose that d* ε D is a Bayes' estimator against a prior distribution ξ*(θ) on the parameter space Θ and the risk function R(d*, θ) = constant on Θ; then d* is a minimax estimator of θ.

The major result of this paper is contained in this theorem and its discussion is given below.

2 MAJOR RESULTS

Theorem 2.1: Let X = ($X_1, X_2, \ldots, X_n$) be 'n' independently and identically distributed random variables drawn from the density (1.1). Then $\hat{\theta}_{MQL} = \dfrac{\Gamma\frac{2n+c}{2}}{\Gamma\frac{2n+c+1}{2}}\left(\frac{S^2}{2}\right)^{1/2}$ is the minimax estimator of the parameter θ for the quadratic loss function (QLF) of the type:

$$L(\theta, d) = \left(\frac{\theta - d}{\theta}\right)^2 \tag{2.1}$$

where θ is the parameter to be estimated and d is the estimate of θ.

Note that a loss function of the form L (θ,d) = $(\theta - d)^2$ is not a minimax estimator for the above distribution.

First we have to prove Theorem 2.1. We use Lehmann's Theorem, which was stated before. Here, we consider the quadratic loss function (QLF) of the form

$$L(\theta, d) = \left(\frac{\theta - d}{\theta}\right)^2$$

which is a non-negative symmetric and continuous loss function of θ and d. In order to prove the theorem, it will be sufficient to show that

$$d = \frac{\Gamma\frac{2n+c}{2}}{\Gamma\frac{2n+c+1}{2}}\left(\frac{S^2}{2}\right)^{1/2} \tag{2.2}$$

is a minimax estimator of θ for the loss function (2.1). For this, first we have to find the Bayes' estimator d of θ. Then, if we can show that the risk function of d is constant, the Theorem 2.1 will be followed. Let us assume that θ has non-informative prior density defined as

$$g(\theta) \propto \frac{1}{\theta^c} \quad ; \theta > 0, c > 0 \tag{2.3}$$

When c =3, we get the ALI prior for the Rayleigh pdf (1.1) because of Hartigan (1964).

As pointed out in (1.1), the likelihood function of the distribution of f(x| θ) is given by

$$L(x_1, x_2, \ldots, x_n \mid \theta) = \left(\prod_{i=1}^{n} x_i\right)\theta^{-2n} e^{-\frac{S^2}{2\theta^2}} \tag{2.4}$$

Assuming that the parameter θ is unknown, the maximum likelihood estimate (MLE) of the parameter θ can be shown to be

$$\hat{\theta}_{MLE} = \left(\frac{S^2}{2n}\right)^{1/2}$$

Combining the likelihood function (2.4) and the prior g(θ) in (2.3), the posterior distribution of θ via Bayes' Theorem for the given random sample X = ($x_1, x_2, \ldots, x_n$) is

$$\pi(\theta \mid x) = \frac{L(x \mid \theta) g(\theta)}{\int_{\Omega} L(x \mid \theta) g(\theta) d\theta} = \frac{\prod_{i=1}^{n} x_i \; \theta^{-2n-c} \, e^{-\frac{s^2}{2\theta^2}}}{\int_0^{\infty} \prod_{i=1}^{n} x_i \; \theta^{-2n-c} \, e^{-\frac{s^2}{2\theta^2}} d\theta}$$

On simplification, we get,

$$= \frac{2\left(\frac{s^2}{2}\right)^{\frac{2n+c-1}{2}} e^{-\frac{s^2}{2\theta^2}}}{\Gamma\left(\frac{2n+c-1}{2}\right) \theta^{2n+c}} \tag{2.5}$$

Now, for the QLF (2.1), the Bayes' estimator of θ is given by

$$d = \frac{\int_0^\infty \frac{1}{\theta}\pi(\theta \mid x)\,d\theta}{\int_0^\infty \frac{1}{\theta^2}\pi(\theta \mid x)\,d\theta} = \frac{\int_0^\infty \frac{1}{\theta^{2n+c+1}} e^{-\frac{s^2}{2\theta^2}}\,d\theta}{\int_0^\infty \frac{1}{\theta^{2n+c+2}} e^{-\frac{s^2}{2\theta^2}}\,d\theta}$$

On simplification, we get,

$$d = \frac{\Gamma\frac{2n+c}{2}}{\Gamma\frac{2n+c+1}{2}} \left(\frac{S^2}{2}\right)^{1/2} \;;\; S^2 = \sum_{i=1}^{n} x_i^2 \tag{2.6}$$

The risk function of the estimator d is

$$R(\theta) = E[\,L(\theta \mid d\,)]$$

$$= \frac{1}{\theta^2}[\theta^2 - 2\theta E(d) + E(d^2)]$$

$$= \frac{1}{\theta^2}\left[\theta^2 - 2\theta \frac{\Gamma(\frac{2n+c}{2})}{\Gamma(\frac{2n+c+1}{2})} \frac{1}{\sqrt{2}} E(s) + \left(\frac{\Gamma(\frac{2n+c}{2})}{\Gamma(\frac{2n+c+1}{2})}\right)^2 \frac{1}{2} E(s^2)\right]$$

$$= \left[1 - 2\sqrt{2n}\, \frac{\Gamma(\frac{2n+c}{2})}{\Gamma(\frac{2n+c+1}{2})} \frac{\Gamma(n+\frac{1}{2})}{\Gamma n} + n\left(\frac{\Gamma(\frac{2n+c}{2})}{\Gamma(\frac{2n+c+1}{2})}\right)^2\right] \tag{2.7}$$

which is constant.

Therefore, according to Lehmann's Theorem, it follows that

$d = \hat{\theta}_{MQL} = \frac{\Gamma\frac{2n+c}{2}}{\Gamma\frac{2n+c+1}{2}}\left(\frac{S^2}{2}\right)^{1/2}$ is the minimax estimator of the parameter θ of the Rayleigh distribution under QLF of the form (2.1).

3 EMPIRICAL STUDY

Mean Squared Errors (MSEs) are considered to compare the different estimators of the parameter θ of the Rayleigh distribution and are obtained by the Maximum likelihood and Minimax for Quadratic Loss function methods. The MSE of an estimator is defined by

$$MSE(\hat{\theta}) = E[(\theta - \hat{\theta})]^2 = Var(\hat{\theta}) + [Bias(\hat{\theta})]^2$$

The estimated values of the parameter and MSE of the estimators are compared by the Monte-Carlo Simulation Method, using the Rayleigh distribution.

Table 1. Estimated values and MSEs of different estimators for the parameter θ of the Rayleigh distribution when θ =1 and c =1

Sample Size	Criteria	$\hat{\theta}_{MQL}$	$\hat{\theta}_{\mathrm{MLE}}$
5	Estimated Value	0.9504	0.9744
	MSE	7.7187	7.8239
10	Estimated Value	0.9696	0.9818
	MSE	0.7906	0.7908
20	Estimated Value	0.9863	0.9924
	MSE	0.1006	0.1008
25	Estimated Value	0.9876	0.9925
	MSE	0.0567	0.0568
30	Estimated Value	0.9915	0.9957
	MSE	0.0335	0.0336

Table 2. Estimated values and MSEs of different estimators for the parameter θ of the Rayleigh distribution when θ =1 and c = -1

Sample Size	Criteria	$\hat{\theta}_{MQL}$	$\hat{\theta}_{\mathrm{MLE}}$
5	Estimated Value	1.0547	0.9732
	MSE	9.8513	8.1132
10	Estimated Value	1.0250	0.9860
	MSE	0.8128	0.7394
20	Estimated Value	1.0169	0.9977
	MSE	0.1101	0.1039
25	Estimated Value	1.0156	1.0003
	MSE	0.0563	0.0534
30	Estimated Value	1.0139	1.0012
	MSE	0.0309	0.0295

Table 3. Estimated values and MSEs of different estimators for the parameter θ of the Rayleigh distribution when θ =1 and c =2

Sample Size	Criteria	$\hat{\theta}_{MQL}$	$\hat{\theta}_{\mathrm{MLE}}$
5	Estimated Value	0.8984	0.9639
	MSE	7.8057	7.3995
10	Estimated Value	0.9416	0.9764
	MSE	0.9382	0.9051
20	Estimated Value	0.9661	0.9841
	MSE	0.1076	0.1043
25	Estimated Value	1.0067	1.0048
	MSE	0.0416	0.0415
30	Estimated Value	0.9854	0.9976
	MSE	0.0211	0.0209

Table 4. Estimated values and MSEs of different estimators for the parameter θ of the Rayleigh distribution when θ =1 and c = -2

Sample Size	Criteria	$\hat{\theta}_{MQL}$	$\hat{\theta}_{\mathrm{MLE}}$
5	Estimated Value	1.1154	0.9670
	MSE	10.4373	6.5056
10	Estimated Value	1.0462	0.9788
	MSE	0.9977	0.8197
20	Estimated Value	1.0150	0.9828
	MSE	0.0977	0.0922
25	Estimated Value	1.0123	0.9867
	MSE	0.0422	0.0403
30	Estimated Value	1.0088	0.9877
	MSE	0.0285	0.0276

Table 5. Estimated values and MSEs of different estimators for the parameter θ of the Rayleigh distribution when θ =1 and c = 3

Sample Size	Criteria	$\hat{\theta}_{MQL}$	$\hat{\theta}_{\mathrm{MLE}}$
5	Estimated Value	0.8675	0.9702
	MSE	8.0569	6.9153
10	Estimated Value	0.9285	0.9850
	MSE	0.9446	0.8789
20	Estimated Value	0.9564	0.9859
	MSE	0.1066	0.0989
25	Estimated Value	0.9793	1.0035
	MSE	0.0489	0.0492
30	Estimated Value	0.9855	1.0058
	MSE	0.0265	0.0270

Table 6. Estimated values and MSEs of different estimators for the parameter θ of the Rayleigh distribution when θ =1 and c = -3

Sample Size	Criteria	$\hat{\theta}_{MQL}$	$\hat{\theta}_{\text{MLE}}$
5	Estimated Value	1.2062	0.9739
	MSE	17.9522	7.6486
10	Estimated Value	1.0880	0.9885
	MSE	1.0695	0.6739
20	Estimated Value	1.0465	0.9997
	MSE	0.1226	0.0963
25	Estimated Value	1.0375	1.0006
	MSE	0.0591	0.0484
30	Estimated Value	1.0353	1.0047
	MSE	0.0306	0.0248

Figure 1:Graph of MSE for different values of n under MQL and MLE when Theta=1 and C=1

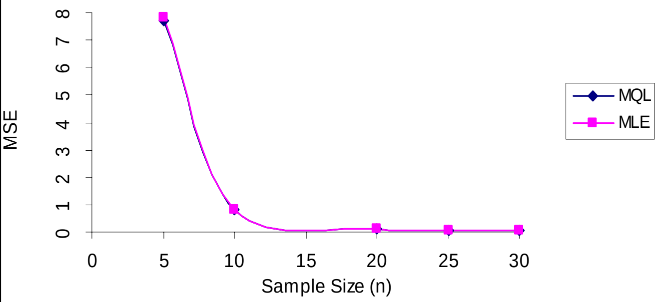
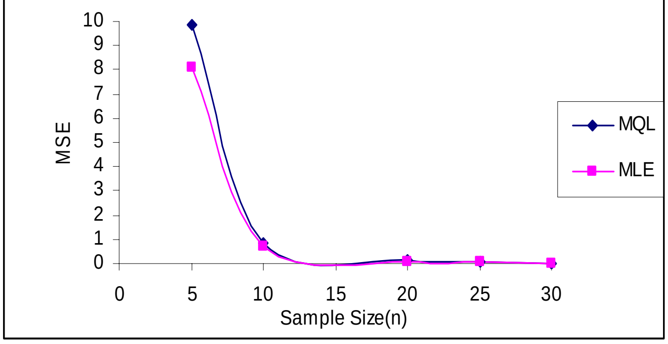


Figure 2:Graph of MSE for different values of n under MQL and MLE when Theta =1and C= - 1



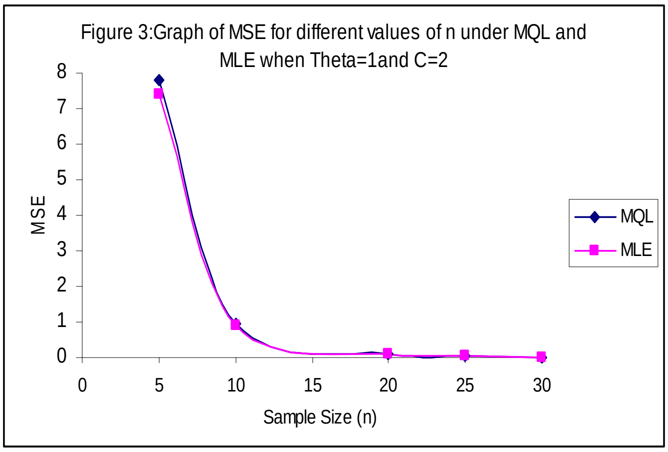


Figure 3:Graph of MSE for different values of n under MQL and MLE when Theta=1and C=2

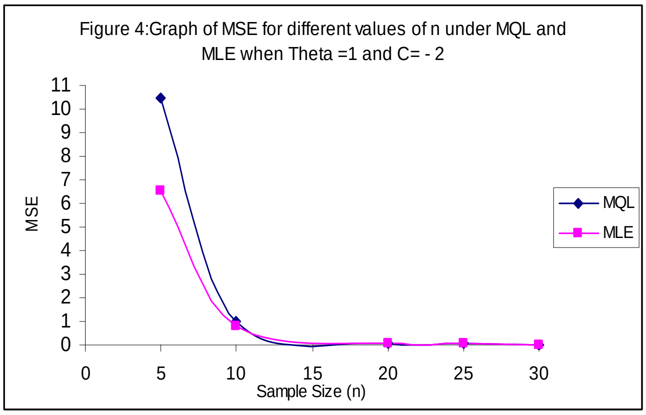


Figure 4:Graph of MSE for different values of n under MQL and MLE when Theta =1 and C= - 2

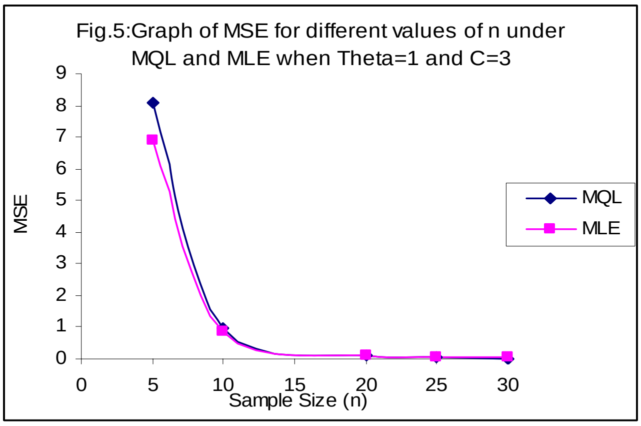
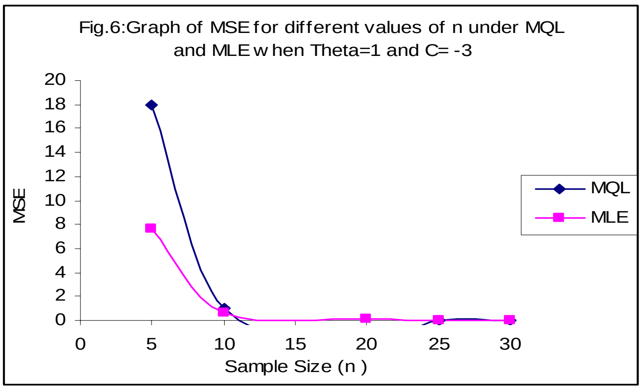


Fig.5:Graph of MSE for different values of n under MQL and MLE when Theta=1 and C=3



4 CONCLUSION

It can be seen from Tables 1, 3, and 5, along with Figures 1, 3 and 5, that the minimax estimator under squared error loss function and the classical maximum likelihood estimator have approximately the same MSEs when the value of '*c*' is positive and sample sizes n>30. Also, it can be seen from Tables 2, 4, and 6, along with Figures 2, 4, and 6, that for small as well as for large sample sizes, the classical maximum likelihood estimator appears to be better than that of minimax estimator under quadratic loss function when the value of **'***c***'** is negative. It is also to be noted that Hartigan's prior gives better results than Jeffrey's prior when $n \geq 25$.

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